Some maths and physics around Laplace equation: Grenn's function, Poisson's equation, Biot-savart law

Emmanuel Branlard

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Chapter 1

Biot-Savart law

1.1 Mathematics

1.1.1 Mathematical equations

Lagrangian Δ is the Laplace operator, which takes the following form in cartesian coordinates:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.1}$$

SEE Operators and differential calculus[1]

Laplace's equation

$$\Delta\varphi = 0 \qquad \text{or} \qquad \nabla^2\varphi = 0 \tag{1.2}$$

Poisson's equation

$$\Delta\varphi = f \tag{1.3}$$

Screened Poisson equation

$$\left[\Delta - \lambda^2\right]\varphi(\mathbf{r}) = -f(\mathbf{r}) \tag{1.4}$$

where λ is a constant, f is an arbitrary function of position (known as the "source function") and φ is the function to be determined. This equation is defined in unbounded space and is subject to the condition that $\varphi(r)$ vanishes sufficiently rapidly as $r \to \infty$.

Elliptic equations Laplace's equation and Poisson's equation are the simplest examples of elliptic partial differential equations. Solutions of Laplace's equation are called harmonic functions.

Short biography -(1749-1827) Pierre-Simon marquis de Laplace: French mathematician and astronomer
-(1781-1840) Simeon-Denis Poisson: French mathematician, geometer and physicist
-() Green

1.1.2 The use of Green function for solving differential equations

Green's function $G(\mathbf{x}, \xi)$ is defined as a integral kernel of a linear operator which inverts a differential operator. It can be defined thus as:

$$\mathcal{L}G(\mathbf{x}, \xi) + \delta(\mathbf{x} - \xi) = 0 \quad x, \xi \in \mathbb{R}^n \tag{1.5}$$

Where $\mathcal{L}$ is a linear differential operator, ξ is an arbitrary point in $\mathbb{R}^n$ and δ is Dirac's delta function. The Green function can be used to solve (weakly) a differential equation of the form:

$$\mathcal{L}u(\mathbf{x}) = \psi(\mathbf{x}) \tag{1.6}$$

If the differential equation is accompanied by appropriate boundary conditions, and if the green function corresponding to $\mathcal{L}$ is known the solution of (1.6) has the following integral representation:

$$u(\mathbf{x}) = -\int_\Omega G(\mathbf{x}, \xi)\, \psi(\xi)\, d\xi \tag{1.7}$$

The demonstration is as follow. Multiplying equation (1.5) by $\psi(\xi)$ and integrating over the bounded space Ω with respect to ξ we have:

$$\int_\Omega \delta(\mathbf{x} - \xi)\, \psi(\xi)\, d\xi = -\int_\Omega \mathcal{L}G(\mathbf{x}, \xi)\, \psi(\xi)\, d\xi \tag{1.8}$$

Due to the Dirac delta function property, the first term is simply:

$$\int_\Omega \delta(\mathbf{x} - \xi)\, \psi(\xi)\, d\xi = \psi(\mathbf{x}) \tag{1.9}$$

Or, recalling the differential equation definition (1.6):

$$\int_\Omega \delta(\mathbf{x} - \xi)\, \psi(\xi)\, d\xi = \mathcal{L}u(\mathbf{x}) \tag{1.10}$$

Hence, equation (1.8) may be rewritten as:

$$\mathcal{L}u(\mathbf{x}) = -\int_\Omega \mathcal{L}G(\mathbf{x}, \xi)\, \psi(\xi)\, d\xi \tag{1.11}$$

Since $\mathcal{L}$ is a linear differential operator, which does not act on the variable of integration we may write:

$$u(\mathbf{x}) = -\int_\Omega G(\mathbf{x}, \xi)\, \psi(\xi)\, d\xi \tag{1.12}$$

Which is the integral representation presented above for $u(x)$. However, to evaluate this integral knowledge of the explicit form of both G and ψ are required. Furthermore, even if both G and ψ are known, the associated integral may not be a trivial exercise. In addition, every linear differential operator does not admit a Green's Function. It would also be prudent to point out that in general, Green's functions are distributions rather than classical functions.

1.1.3 Resolution of poisson screened equation with the use of Fourier transform

The resolution of equation (1.4) is performed through a 3d spacial Fourier transfomation of function $f(r)$, this operation being possible due to the unbounded domain of definition $-\infty < x, y, z < \infty$. The Fourier transform and its inverse are respectivly defined as:

$$\hat{f}(\boldsymbol{k}) \quad = \mathcal{F}(f) \quad = \int_{\Omega} f(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3 r \tag{1.13}$$

$$f(\boldsymbol{r}) \quad = \mathcal{F}^{-1}(f) \quad = \frac{1}{(2\pi)^3} \int_{\hat{\Omega}} \hat{f}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3 k \tag{1.14}$$

$$\tag{1.15}$$

Note: multiple conventions exists for Fourier transform, one could have used a coefficient $1/(2\pi)^{3/2}$ before the integral for both the Fourier transform and its inverse, or no coefficient at all but an exponential coefficient with $2\pi i$. The choice made here is justified by the normalization convention $\mathcal{F}(\delta(\boldsymbol{r})) = 1$

The Fourier transform of equation (1.4), after two successive integrations and the use of boundary condition, reduces eventually to the algebraic equation:

$$\left(k^2 + \lambda^2\right) \hat{u}(\boldsymbol{k}) = \hat{f}(\boldsymbol{k}) \tag{1.16}$$

Which yields the solution:

$$\hat{u}(\boldsymbol{k}) = \frac{\hat{f}(\boldsymbol{k})}{k^2 + \lambda^2} \tag{1.17}$$

The Fourier inverse transformation provides the desired solution:

$$u(\boldsymbol{r}) = \mathcal{F}^{-1}(u) = \frac{1}{(2\pi)^3} \int \frac{\hat{f}(\boldsymbol{k})}{k^2 + \lambda^2} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3 k \tag{1.18}$$

The integrand is known because f is known and it is straightforward to compute $\hat{f}$. Thus one can be satisfied of this expression. Nevertheless it is interesting to develop the expression of $\hat{f}(\boldsymbol{k})$ as a Fourier transform, leading to the double integral:

$$u(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int\int \frac{1}{k^2 + \lambda^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} f(\boldsymbol{r}') d^3 r' d^3 k \tag{1.19}$$

Expression in which we recognize an integral representation for the Green's function G for equation (1.4):

$$u(\boldsymbol{r}) = \int G(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}') d^3 r' \tag{1.20}$$

with

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{(2\pi)^3} \int \frac{1}{k^2 + \lambda^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})} d^3 k \tag{1.21}$$

The formalism of Green function is common in differential equation formalism and deserved attention. This approach will be thus studied in the next subsection.

1.1.4 Resolution of poisson screened equation with the use of Green function

Poisson's screened equation uses a linear operator so its solutions are superposable and this suggests a general method for solving this equation. Suppose that we could construct all of the solutions generated by point sources, provided that they satisfy the appropriate boundary conditions. Any general source function can be built up out of a set of suitably weighted point sources, so the general solution of Poisson's equation must be expressible as a weighted sum over the point source solutions. Thus, once we know all of the point source solutions we can construct any other solution. In mathematical terminology, we require the solution to

$$\triangle G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{1.22}$$

which goes to zero as $|\boldsymbol{r}| \to \infty$. The function $G(\boldsymbol{r}, \boldsymbol{r}')$ is the solution generated by a unit point source located at position $\boldsymbol{r}'$. This function is known as a Green's function and from now on we will consider it to depend only on the difference $\boldsymbol{r} - \boldsymbol{r}$. In other words

$$G(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r} - \boldsymbol{r}') \tag{1.23}$$

This reflects the translational invariance of the unbounded domain with the disturbance depending only on the relative separation from the source. The solution generated by a general source function $f(\boldsymbol{r})$ is simply the appropriately weighted sum of all of the Green's function solutions:

$$u(\boldsymbol{r}) = \int_\Omega G(\boldsymbol{r} - \boldsymbol{r}') f(\boldsymbol{r}') d^3 r'. \tag{1.24}$$

The formalism of Green function and the proof of this solution is provided in annex 1.1.2. As stated above, this is a superposition of screened $1/r$ functions, weighted by the source function f and with λ acting as the strength of the screening.

An analytical expression of the Green functions associated to Poisson's screened equation can be evaluated. The presence of the term k^2 suggests the use of spherical polar coordinates (ρ, θ, ϕ), with the polar axis along $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, such that, $\boldsymbol{p} = \boldsymbol{\rho}$, so $\boldsymbol{\rho} \cdot \boldsymbol{R} = \rho R \cos\theta$, and the elementary integration volume being $dV = d\rho\, \rho d\theta\, \rho \sin\theta d\phi$. Equation (1.21) becomes:

$$G(\boldsymbol{R}) = \frac{1}{(2\pi)^3} \int_0^\infty \rho^2 \int_0^\pi \sin\theta \frac{e^{i\rho R\cos\theta}}{\rho^2 + \lambda^2} d\theta d\rho \int_0^{2\pi} d\phi \tag{1.25}$$

The angular integration present no difficulty, letting us with the ρ integral:

$$G(\boldsymbol{R}) = \frac{1}{2\pi^2 R} \int_0^\infty \frac{\rho}{\rho^2 + \lambda^2} \frac{e^{i\rho R} - e^{-i\rho R}}{2i} d\rho \tag{1.26}$$

And the beauty of holomorph analysis to gives us:

$$G(\boldsymbol{R}) = \frac{1}{4\pi^2 R} \int_{-\infty}^\infty \frac{i\rho}{\rho^2 + \lambda^2} e^{i\rho R} d\rho = \frac{1}{4\pi^2 R} \cdot \left[2\pi i \operatorname{Res}\left(\frac{i\rho e^{i\rho R}}{(\rho + i\lambda)(\rho - i\lambda)}, i\lambda \right) \right] = \frac{e^{-\lambda R}}{4\pi R} \tag{1.27}$$

with the use of contour techniques around the pole $\rho = i\lambda$ and the residue theorem.

Let's summarize the previous steps and write the final results for the particular case of interest where $\lambda = 0$, corresponding to the solution of Poisson's equation:

$$G(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{(2\pi)^3} \int \frac{1}{k^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}' - \boldsymbol{r})} d^3 k = \frac{1}{4\pi |\boldsymbol{r}' - \boldsymbol{r}|} \tag{1.28}$$

$$\varphi(\boldsymbol{r}) = \int_\Omega f(\boldsymbol{r}') G(\boldsymbol{r} - \boldsymbol{r}') dr' = f * G = \frac{1}{4\pi} \int_\Omega \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} dr' \tag{1.29}$$

The expression of φ as the convolution between f and G is an interesting results for implementing Poisson's solver algorithm.

1.2 Electromagnetism

The resolution of Poisson equation is the usual way to find the electric potential for a given charge distribution.

1.2.1 Poisson equation

Assuming a linear, isotropic and homogeneous medium, Gauss' law is written as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon} \tag{1.30}$$

In the absence of a changing magnetic field, $\boldsymbol{B}$, Faraday's law of induction gives:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{1.31}$$

Since the curl of the electric field is zero, it is defined by a scalar electric potential field ϕ:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi \tag{1.32}$$

Substituting $\boldsymbol{E}$ provides us with a form of the Poisson equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \boldsymbol{\nabla}^2\phi = \Delta\phi = -\frac{\rho}{\epsilon}. \tag{1.33}$$

Equation (1.33) is a particular case of the inhomogeneous partial differential equation named Screened Poisson equation which with usual mathematical notation is the following:

Let's summarize the previous steps and write the final results for the particular case of interest where $\lambda = 0$, corresponding to the solution of Poisson's equation:

$$G(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{(2\pi)^3} \int \frac{1}{k^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})} d^3k = \frac{1}{4\pi|\boldsymbol{r'} - \boldsymbol{r}|} \tag{1.34}$$

$$\phi(\boldsymbol{r}) = \frac{1}{\epsilon_0} \int_\Omega \rho(\boldsymbol{r'}) G(\boldsymbol{r} - \boldsymbol{r}') dr' = \frac{1}{\epsilon_0} \rho * G = \frac{1}{4\pi\epsilon_0} \int_\Omega \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r'} - \boldsymbol{r}|} dr' \tag{1.35}$$

The expression of ϕ as the convolution between ρ and G is an interesting results for implementing Poisson's solver algorithm.

1.3 Fluid Mechanics

1.3.1 Velocity induced by a vorticity field in an incompressible flow

Let Ω be the bound domain considered, and $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{U}$ be a given vorticity field, non zero, such that $\operatorname{div} \boldsymbol{\omega} = 0$. The vorticity field being given, the problem consists in finding the velocity field satisfying the two equations:

$$\operatorname{curl} \boldsymbol{U} = \boldsymbol{\omega} \tag{1.36}$$

$$\operatorname{div} \boldsymbol{U} = 0 \tag{1.37}$$

The system above, is an inhomogeneous system. A particular solution of this system will be found below, but a general solution of the homogeneous system should be added to solve a particular boundary condition problem.

Particular solution of the inhomogeneous system Since "div curl $\equiv 0$", Eq. (1.37) allows to write $\boldsymbol{U}$ as

$$\boldsymbol{U} = \operatorname{curl} \boldsymbol{A} \tag{1.38}$$

where the potential field A can be ensured to be unique by adding the condition!!! $\operatorname{div} \boldsymbol{A} = 0$. Inserting Eq. (1.38) into Eq. (1.36) and making use of the relation "curl curl = grad div $-\Delta$" leads to:

$$\operatorname{curl}(\operatorname{curl} \boldsymbol{A}) = \boldsymbol{\omega} \tag{1.39}$$

$$\operatorname{grad}(\operatorname{div} \boldsymbol{A}) - \Delta \boldsymbol{A} = \boldsymbol{\omega} \tag{1.40}$$

$$\Rightarrow \quad \Delta \boldsymbol{A} = -\boldsymbol{\omega} \tag{1.41}$$

Each component of $\boldsymbol{A}$ hence satisfies a given form of Poisson's equation with different left hand side which are the corresponding component of the vector $\boldsymbol{\omega}$, i.e., for $i = 1, 2, 3$, $\Delta A_i = -\omega_i$, which according to ???? are solved as $A_i = -\omega_i * G$ and can be summarized in a vectorial way as:

$$\boldsymbol{A}(\boldsymbol{r}) = -\boldsymbol{\omega} * G = \frac{1}{4\pi} \int_\Omega \frac{\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, dv(\boldsymbol{r}') \tag{1.42}$$

This solution verifies $\operatorname{div} \boldsymbol{A} = 0$ since $\operatorname{div} \boldsymbol{\omega} = 0$, and is hence the solution of the problem that was sought. The velocity field $\boldsymbol{U} = \operatorname{curl} \boldsymbol{A}$ is from Eq. (1.42):

$$\boldsymbol{U}(\boldsymbol{r}) = \frac{1}{4\pi} \operatorname{curl}_{\boldsymbol{r}} \left[\int_\Omega \frac{\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, dv(\boldsymbol{r}') \right] \tag{1.43}$$

$$\boldsymbol{U}(\boldsymbol{r}) = \frac{1}{4\pi} \int_\Omega \operatorname{curl}_{\boldsymbol{r}} \left[\frac{\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] dv(\boldsymbol{r}') \tag{1.44}$$

Using the relation "$\operatorname{curl}(a\boldsymbol{b}) = a \operatorname{curl}(\boldsymbol{b}) + \operatorname{grad}(a) \times \boldsymbol{b}$" the term in the integral develops as:

$$\operatorname{curl}_{\boldsymbol{r}} \left[\frac{\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \operatorname{curl}_{\boldsymbol{r}} \boldsymbol{\omega}(\boldsymbol{r}') + \operatorname{grad}_{\boldsymbol{r}} \left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] \times \boldsymbol{\omega}(\boldsymbol{r}') \tag{1.45}$$

$$= \operatorname{grad}_{\boldsymbol{r}} \left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] \times \boldsymbol{\omega}(\boldsymbol{r}') \tag{1.46}$$

$$= -\frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \times \boldsymbol{\omega}(\boldsymbol{r}') \tag{1.47}$$

Hence the velocity field $\boldsymbol{U}$ induced by a given vorticity field $\boldsymbol{\omega}$ in an incompressible flow is:

$$\boldsymbol{U}(\boldsymbol{r}) = -\frac{1}{4\pi} \int_\Omega \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \times \boldsymbol{\omega}(\boldsymbol{r}') \, dv(\boldsymbol{r}') \tag{1.48}$$

General solution of the homogeneous system The homogeneous system corresponding to the system Eq. (1.36) and (1.37) is:

$$\operatorname{curl} \boldsymbol{U} = 0 \tag{1.49}$$

$$\operatorname{div} \boldsymbol{U} = 0 \tag{1.50}$$

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1.4 Biot-Savart law in term of solid angle for a closed path

Lets consider a closed path, on which the problem quantity f is known and constant all along this path. In electromagnetism, this would be a circuit with constant intensity I, and in fluid mechanics a closed vortex ring of intensity $\boldsymbol{gamma}$. The Biot-Savart law writes in this case

$$\boldsymbol{Q}(\boldsymbol{r}) = \frac{f}{4\pi} \operatorname{curl} \oint \frac{\boldsymbol{dl} r'}{|\boldsymbol{r} - \boldsymbol{r'}|} = \frac{f}{4\pi} \operatorname{curl} \boldsymbol{R} \tag{1.51}$$

where $\boldsymbol{R}$ has been introduced for convenience. Integral of scalar quantities on closed path can be expressed in terms of surface by vertue of Stoke's theorem:

$$Sokes \tag{1.52}$$

The idea is then to use Stokes theorem on each component of the vector $\boldsymbol{R}$. Writing $\boldsymbol{e_i}$ the unitary vector along the i^{th} axis, then the i^{th} component of $\boldsymbol{R}$ is:

$$R_i = \oint \frac{\boldsymbol{dl} r' \cdot \boldsymbol{e_i}}{|\boldsymbol{r} - \boldsymbol{r'}|} \tag{1.53}$$

If S is now any surface bounded by the path C, then by application of Stoke's theorem:

$$R_i = \oint \frac{\boldsymbol{dl} r' \cdot \boldsymbol{e_i}}{|\boldsymbol{r} - \boldsymbol{r'}|} = \int_S \operatorname{curl}_{\boldsymbol{r'}} \left[\frac{\boldsymbol{e_i}}{|\boldsymbol{r} - \boldsymbol{r'}|} \right] \cdot \boldsymbol{n}(\boldsymbol{r'}) \, dS \tag{1.54}$$

Once again, using the relation "$\operatorname{curl}(a\boldsymbol{b}) = a \operatorname{curl}(\boldsymbol{b}) + \operatorname{grad}(a) \times \boldsymbol{b}$" and the triple product "$\boldsymbol{n} \cdot [\boldsymbol{b} \times \boldsymbol{c}] = \boldsymbol{c} \cdot [\boldsymbol{n} \times \boldsymbol{b}]$", the above equation writes:

$$R_i = \int_S \left[\boldsymbol{n}(\boldsymbol{r'}) \times \operatorname{grad}_{\boldsymbol{r'}} \frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} \right] \cdot \boldsymbol{e_i} \, dS \tag{1.55}$$

The scalar product with $\boldsymbol{e_i}$ allows to write a vectorial expression for $\boldsymbol{R}$:

$$\boldsymbol{R} = \int_S \boldsymbol{n}(\boldsymbol{r'}) \times \operatorname{grad}_{\boldsymbol{r'}} \frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} \, dS \tag{1.56}$$

Given the symmetry between $\boldsymbol{r}$ $\boldsymbol{r'}$ in the expression of the gradient, the gradient can be expressed in terms of $\boldsymbol{r}$. Using one last time the relation "$\operatorname{curl}(a\boldsymbol{b}) = a \operatorname{curl}(\boldsymbol{b}) + \operatorname{grad}(a) \times \boldsymbol{b}$", leads to:

$$\boldsymbol{R} = \operatorname{curl} \int_S \frac{\boldsymbol{n}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} = \operatorname{curl} \boldsymbol{W} \tag{1.57}$$

Where $\boldsymbol{W}$ is introduced to simplify notations. Going back to this expression of $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \frac{f}{4\pi} \operatorname{curl} [\operatorname{curl} \boldsymbol{W}] \tag{1.58}$$

The relation "$\operatorname{curl} \operatorname{curl} = \operatorname{grad} \operatorname{div} - \Delta$" leads to:

$$\operatorname{curl} \operatorname{curl} \boldsymbol{W} = \operatorname{grad} \left[\int_S -\frac{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, dS \right] - \left[-4\pi \int_S \boldsymbol{n} \delta(\boldsymbol{r} - \boldsymbol{r'}) \, dS \right] \tag{1.59}$$

$$= -\operatorname{grad} \boldsymbol{\Omega} + 0 \tag{1.60}$$

with Ω being the solid angle defined as:

$$\Omega = \int_S -\frac{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, dS \tag{1.61}$$

And the Biot-Savart law can be expressed for the field $\boldsymbol{Q}$ generated by a closed path of constant quantity f as:

$$\boldsymbol{Q} = -1 \frac{f}{4\pi} \operatorname{grad} \Omega \tag{1.62}$$

Bibliography

[1] E. Branlard. Useful relations of differential calculus. *http://emmanuel.branlard.free.fr*, 2012.