Differential calculus

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Chapter 1

Differential Calculus

1.1 Differentiation - DL - Taylor

$$f(x,t+dt) = f(x,t) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + O(dt^2) \tag{1.1}$$

$$f(x+dx,t) = f(x,t) + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + O(dx^2) \tag{1.2}$$

$$f(x+dx,t+dt) = f(x,t) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dx + \frac{\partial^2 f}{\partial x \partial t}dxdt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + O(dx^2, dt^2) \tag{1.3}$$

1.2 Tensors

Einstein's convention Einstein convention (): implicitly do the sum from 1 to n, where n is the dimension of the vectorial space considered, on each index that are repeated at least once per "momome" and on two different "holders". In an orthonormal base $\boldsymbol{U} \cdot \boldsymbol{V} = U_i V_i$, $\|\boldsymbol{U}\|^2 = U_i U_i$

In non orthonormal base, a vector can be decomposed on contravariant X^i and covariant coordinates X_i. The covariante coordinates are related to the dual space. $\boldsymbol{X} = X^i \boldsymbol{e}_i$

Tensors A tensor of order p is a a p-linear form on $\underbrace{E \times \cdots \times E}_{p}$ Tensors of order 1 corresponds to vectors of E [1] and wil be written $\boldsymbol{U}$. Tensors of order 2 are the bilinear form on $E \times E$, which have a one-to-one relation to linear applications from E to E(endomorphism). From this isomorphism between $\mathcal{L}(E \times E, \mathbb{R})$ and $\mathcal{L}(E, E)$, for a given base tensors can be written in matricial form. Tensors of order 2 will be written $\underline{\underline{\boldsymbol{T}}}$.

$$\underline{\underline{\boldsymbol{T}}}(\boldsymbol{U}, \boldsymbol{V}) = U_i V_j \underline{\underline{\boldsymbol{T}}}(\boldsymbol{e}_i, \boldsymbol{e}_j) = U_i T_{ij} V_j \tag{1.4}$$

It is emphasized that that tensors are independent of the basis of the vectorial space. The basis

[1] This is due to the isomorphism between the dual space E^* consisting of the linear form on E and E. All linear form from E^* can be written as a scalar product, and hence be associated to a vector in E

only comes to play when a numerical representation of the tensor is sought:

$$\underbrace{\boldsymbol{U}}_{\text{independent of basis}} = \underbrace{U_i \boldsymbol{e}_i}_{\text{dependent on the basis}} \tag{1.5}$$

Tensorial product Tensorial product of order 1 tensors:

$$\begin{aligned} \boldsymbol{U} \otimes \boldsymbol{V} \ : \quad & E \longrightarrow E \\ & \boldsymbol{V} \longmapsto (\boldsymbol{V} \cdot \boldsymbol{X}) \boldsymbol{U} \end{aligned}$$

In particular:

$$(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)(\boldsymbol{e}_k) = (\boldsymbol{e}_j \cdot \boldsymbol{e}_k)\boldsymbol{e}_i = \delta_{jk}\boldsymbol{e}_i \tag{1.6}$$

From the definition of the tensorial product of two tensors of first order, it is seen that in this case the operator $\otimes$ is an non symetric bilinear application of $E \times E$ in $\mathcal{L}(E, E)$, and hence is a tensor of second order, hence the following proposition:

$$\text{if } \underline{\underline{\boldsymbol{T}}} = (\boldsymbol{U} \otimes \boldsymbol{V}) \text{ then } \underline{\underline{\boldsymbol{T}}} = U_i V_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = T_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.7}$$

The above is not true reciprocally, a tensor of order 2 is not always written as a tensorial product of two tensor of two tensors of order 1.

$$\underline{\underline{\mathbb{1}}} = \delta_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.8}$$

$$\underline{\underline{\boldsymbol{h}}} = \boldsymbol{f} \otimes \boldsymbol{g} = \begin{pmatrix} f_1 \, g_1 & f_1 \, g_2 & f_1 \, g_3 \\ f_2 \, g_1 & f_2 \, g_2 & f_2 \, g_3 \\ f_3 \, g_1 & f_3 \, g_2 & f_3 \, g_3 \end{pmatrix} \tag{1.9}$$

In general a tensorial product between a tensor of order p and a tensor of order q is a tensor of order $p + q$.

Contraction The contraction between a tensor S of order p and a tensor T of order q is a tensor of order $p + q - 2$ obtained by identification of the last index of S and the first index of T. It is an extension of the scalar product. Examples:

$$\underline{\underline{\underline{S}}} \cdot \underline{\underline{\boldsymbol{T}}} = (S_{ijk} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k) \cdot (T_{pq} \, \boldsymbol{e}_p \otimes \boldsymbol{e}_q) = S_{ijk} T_{kq} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_q \tag{1.10}$$

$$\underline{\underline{\boldsymbol{T}}} \cdot \boldsymbol{U} = (T_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (U_k \boldsymbol{e}_k) = T_{ij} U_j \, \boldsymbol{e}_i \tag{1.11}$$

$$\boldsymbol{U} \cdot \underline{\underline{\boldsymbol{T}}} = (U_i \boldsymbol{e}_i) \cdot (T_{jk} \, \boldsymbol{e}_j \otimes \boldsymbol{e}_k) = U_i T_{ik} \, \boldsymbol{e}_k = T_{ji} U_j \, \boldsymbol{e}_i \tag{1.12}$$

It should be noted that even though Eq. (1.11) recalls the classic-matrix vector multiplication, the contraction has actually a more general context. The contraction from Eq. (1.12) resemble the vector matrix multiplication, but does not require a notion of vector in column.

Double contraction The double contraction consists in performing the contraction twice, and apply hence to tensors satisfying $p + q - 4 \geq 0$. The operator can we written as "$\cdot\cdot$", "$:$" or "$\odot$". An example found in fluid dynamics is the double contraction of tensors of order 2:

$$\underline{\underline{\boldsymbol{S}}} : \underline{\underline{\boldsymbol{T}}} = (S_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (T_{kl} \, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) = S_{ik} T_{ki} \tag{1.13}$$

Propositions Trace of a tensor of order 2: $\mathrm{tr}(\underline{\underline{\boldsymbol{T}}}) = \underline{\underline{\boldsymbol{T}}} : \underline{\underline{\mathbb{1}}}$. Writing this definition in a given base, will lead to the sum of the diagonal elements of the matricial representation of $\underline{\underline{\boldsymbol{T}}}$ in this base. The definition of the trace in tensorial notation hence confirms the results that the trace of the matricial representation of an endomorphism in a given basis is an invariant from one base to another.

The transposed tensor of a tensor $\underline{\underline{\boldsymbol{T}}}$ of order 2 satisfies : ${}^t\underline{\underline{\boldsymbol{T}}}(\boldsymbol{U}, \boldsymbol{V}) = \underline{\underline{\boldsymbol{T}}}(\boldsymbol{V}, \boldsymbol{U})$. Tensors of order 2 can be decomposed into a symmetric and antisymmetric part as:

$$\underline{\underline{\boldsymbol{T}}} = \frac{1}{2}\left(\underline{\underline{\boldsymbol{T}}} + {}^t\underline{\underline{\boldsymbol{T}}}\right) + \frac{1}{2}\left(\underline{\underline{\boldsymbol{T}}} - {}^t\underline{\underline{\boldsymbol{T}}}\right) = \underline{\underline{\boldsymbol{T_s}}} + \underline{\underline{\boldsymbol{T_a}}} \tag{1.14}$$

A tensor can also be decomposed on a spherical and deviation part as :

$$\underline{\underline{\boldsymbol{T}}} = \underline{\underline{\boldsymbol{T}}}^{(s)} + \underline{\underline{\boldsymbol{T}}}^{(d)} \tag{1.15}$$

with $\underline{\underline{\boldsymbol{T}}}^{(s)} = \frac{1}{3}\,\mathrm{tr}(\underline{\underline{\boldsymbol{T}}})\underline{\underline{\mathbb{1}}}$, and $\underline{\underline{\boldsymbol{T}}}^{(d)} = \underline{\underline{\boldsymbol{T}}} - \underline{\underline{\boldsymbol{T}}}^{(s)}$ and hence $\mathrm{tr}(\underline{\underline{\boldsymbol{T}}}) = 0$.

Summary Summary and usual cases in an space with orthonormal base:

$$\boldsymbol{U} \cdot \boldsymbol{V} = (U_i\boldsymbol{e}_i) \cdot (V_j\boldsymbol{e}_j) = U_iV_j(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = U_iV_i \tag{1.16}$$
$$\boldsymbol{U} \otimes \boldsymbol{V} = (U_i\boldsymbol{e}_i) \otimes (V_j\boldsymbol{e}_j) = U_iV_j\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.17}$$
$$\boldsymbol{U} \cdot \underline{\underline{\boldsymbol{T}}} = (U_i\boldsymbol{e}_i) \cdot (T_{jk}\,\boldsymbol{e}_j \otimes \boldsymbol{e}_k) = U_iT_{ik}\,\boldsymbol{e}_k = T_{ji}U_j\,\boldsymbol{e}_i \tag{1.18}$$
$$\underline{\underline{\boldsymbol{S}}} : \underline{\underline{\boldsymbol{T}}} = (S_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (T_{kl}\,\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = S_{ik}T_{ki} \tag{1.19}$$

1.3 Vectorial analysis

Given a vector and a base such that $\boldsymbol{x} = x_i\boldsymbol{e}_i$, the elementary displacement is defined as:

$$\boldsymbol{dx} = d(x_i\boldsymbol{e}_i) = dx_i\boldsymbol{e}_i + x_i d\boldsymbol{e}_i \tag{1.20}$$

Differentiation of a scalar function - introducing the gradient

$$\begin{aligned} \varphi :\quad & E \longrightarrow \mathbb{R} \\ & \boldsymbol{x} \longmapsto \varphi(\boldsymbol{x}) = \varphi(x_1, x_2, \ldots, x_n) \end{aligned}$$

The function φ is a function of several variables, and it's elementary variation at the point $\boldsymbol{x}$ can be written at first order as follow:

$$d\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x} + \boldsymbol{dx}) - \varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x}) + \partial_i\varphi(\boldsymbol{x})\,dx_i + o(\|\boldsymbol{dx}\|) - \varphi(\boldsymbol{x}) \tag{1.21}$$
$$= \partial_i\varphi(\boldsymbol{x})\,dx_i + o(\|\boldsymbol{dx}\|) \tag{1.22}$$

The elementary variation of φ is at first order a linear form with respect to $\boldsymbol{dx}$. This linear form is a tensor of order 1 referred to as the gradient defined as :

$$d\varphi \overset{\sim}{=} \mathrm{grad}\,\varphi \cdot \boldsymbol{dx} \qquad \text{(exact at first order)} \tag{1.23}$$

The above definition is independent of the basis chosen. Example for a three dimensional space with a fixed Cartesian base:

$$dU_x(\boldsymbol{x}) = \mathrm{grad}\,U_x(\boldsymbol{x}) = \frac{\partial U_x}{\partial x}(\boldsymbol{x})dx + \frac{\partial U_x}{\partial z}(\boldsymbol{x})dy + \frac{\partial U_x}{\partial z}(\boldsymbol{x})dz \tag{1.24}$$

If the base is moving one has to use Eq. (1.20), project the moving basis into a fixed one, and use Eq. (1.23) to find the gradient expression by identification.

Further, the derivation with respect to a direction can be defined. If $\boldsymbol{n}$ is a unitary vector, the derivative of ϕ with respect to the direction defined by $\boldsymbol{n}$ at point $\boldsymbol{x}$, is written $\varphi'(\boldsymbol{x},\boldsymbol{n})$ or $\partial\varphi/\partial\boldsymbol{n}$ and defined as:

$$\frac{\partial\varphi}{\partial\boldsymbol{n}} \equiv \varphi'(\boldsymbol{x},\boldsymbol{n}) \triangleq \lim_{h\to 0}\frac{\varphi(\boldsymbol{x}+h\boldsymbol{n})-\varphi(\boldsymbol{x})}{h} = \partial_i\varphi(\boldsymbol{x})\,n_i + o(\|\boldsymbol{n}\|) \tag{1.25}$$

And hence at first order:

$$\frac{\partial\varphi}{\partial\boldsymbol{n}} \equiv \varphi'(\boldsymbol{x},\boldsymbol{n}) \overset{\sim}{=} \operatorname{grad}\varphi(\boldsymbol{x})\cdot\boldsymbol{n} \tag{1.26}$$

It is also interesting to consider the composition of a function with vectorial values $\boldsymbol{V}:\ t\mapsto\boldsymbol{V}(t)$, assuming it's derivability, and the differentiability of φ, the composed function $g:\ t\mapsto\varphi(\boldsymbol{V}(t))$ has also derivability and:

$$\dot{g}(t) = \operatorname{grad}\varphi(\boldsymbol{V}(t))\cdot\dot{\boldsymbol{V}}(t) \tag{1.27}$$

Gradient of a first order tensor The generalisation of the definition of the gradient is done as:

$$d\boldsymbol{U} = \underline{\underline{\operatorname{grad}\boldsymbol{U}}}\cdot\boldsymbol{dX} \tag{1.28}$$

In the general case where the base is not fixed:

$$d\boldsymbol{U} = \partial_i\boldsymbol{U}\,dx_i = \partial_i(U_j\boldsymbol{e}_j)\,dx_i = (\partial_iU_j\,\boldsymbol{e}_j + U_j\underbrace{\partial_j\boldsymbol{e}_i}_{\gamma^k_{ij}\boldsymbol{e}_k})dx_i = \underbrace{\left(\partial_jU_i + U_k\gamma^j_{ki}\right)}_{\left(\underline{\underline{\operatorname{grad}U}}\right)_{ij}}\boldsymbol{e}_j dx_i \tag{1.29}$$

The coordinates of the derivatives of the base coefficient, written γ above, are known as Christoffel's coefficient. In case of a fixed base they are identically zero and the gradient is simply:

$$\underline{\underline{\operatorname{grad}\boldsymbol{U}}} = \partial_jU_i\,\boldsymbol{e_i}\otimes\boldsymbol{e}_j \tag{1.30}$$

Developing these terms in three dimension with a cartesian base leads to the following matricial representation:

$$\underline{\underline{\operatorname{grad}\boldsymbol{U}}} = \underline{\underline{\nabla}}\boldsymbol{U} = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{\partial U_x}{\partial y} & \frac{\partial U_x}{\partial z} \\ \frac{\partial U_y}{\partial x} & \frac{\partial U_y}{\partial y} & \frac{\partial U_y}{\partial z} \\ \frac{\partial U_z}{\partial x} & \frac{\partial U_z}{\partial y} & \frac{\partial U_z}{\partial z} \end{pmatrix} = \begin{pmatrix} {}^t\nabla U_x \\ {}^t\nabla U_y \\ {}^t\nabla U_z \end{pmatrix} \tag{1.31}$$

The transpose notation above assumes a representation of the gradient in column notation, but strictly speaking tensors(and hence the gradient) are independent of the notion of columns and rows introduced in matrix formalism.

Divergence The divergence tensor is defined as the double contraction between the gradient of a tensor of order $p\geq 1$ with the identity tensor of order 2. For instance the divergence of a tensor of order 1:

$$\operatorname{div}\boldsymbol{U} \triangleq \underline{\underline{\operatorname{grad}U}} : \underline{\underline{\mathbb{1}}} = \operatorname{tr}(\underline{\underline{\operatorname{grad}\boldsymbol{U}}}) \tag{1.32}$$

Two definitions of the divergence of a tensor of order 2 is found in the litterature[**?**], Some authors also omit the distinction between using covariante and contravariante coordinates which could lead to confusion. In this document the notation div is consisten with the definition given above. In the book of Cottet[?] a different definition of the divergence is found, and will further be written div$_2$. For a tensor of order two the two definition of the divergence are:

$$\operatorname{div}\underline{\underline{\boldsymbol{T}}} = \partial_j(T_{ij})\boldsymbol{e}_i \tag{1.33}$$
$$\operatorname{div}_2\underline{\underline{\boldsymbol{T}}} = \partial_j(T_{ji})\boldsymbol{e}_i = \partial_i(T_{ij})\boldsymbol{e}_j \tag{1.34}$$

As an illustration of the difference, a useful relation in fluid dynamics is written with these two definitions for two vectors $\boldsymbol{u}$ and $\boldsymbol{\omega}$:

$$\operatorname{div}(\boldsymbol{\omega}\otimes\boldsymbol{u}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\operatorname{div}\boldsymbol{u} \tag{1.35}$$
$$\operatorname{div}_2(\boldsymbol{u}\otimes\boldsymbol{\omega}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\operatorname{div}\boldsymbol{u} \tag{1.36}$$

Above, the definition of the tensorial product is the same, only the definition of the divergence differs. It can be seen that the same result is obtained by inverting the order of the tensorial product but care has to be taken in the notations. It is worth mentionning, that noting $\underline{\underline{\hat{\boldsymbol{T}}}}$ the tensor of $\underline{\underline{\boldsymbol{T}}}$ in covariate coordinates, then if $\boldsymbol{u}$ is a vector written in contravariate coordinates then:

$$\operatorname{div}(\boldsymbol{u}\otimes\underline{\underline{\hat{\boldsymbol{T}}}}) = (\boldsymbol{u}\cdot\nabla)\underline{\underline{\hat{\boldsymbol{T}}}} + \underline{\underline{\hat{\boldsymbol{T}}}}\operatorname{div}\boldsymbol{u} \tag{1.37}$$

For a fix orthonormal basis:

order	tensor	gradient	divergence
0	φ	$\operatorname{grad}\phi$	NA
1	$\boldsymbol{U} = U_i\boldsymbol{e}_i$	$\underline{\underline{\operatorname{grad}\boldsymbol{U}}} = \partial_j U_i\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$	$\nabla\cdot\boldsymbol{U} = \partial_i U_i$
2	$\underline{\underline{\boldsymbol{T}}} = T_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$	$\underline{\underline{\underline{\nabla T}}} = \partial_k T_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k$	$\nabla\cdot\underline{\underline{\boldsymbol{T}}} = \partial_j T_{ij}\,\boldsymbol{e}_i$

Rotational The definition of the rotational can be done in different ways using for instance pseudo-vectors. The rotational of a tensor $\boldsymbol{U}\in\mathbb{R}^3$, noted $\operatorname{curl}\boldsymbol{U}$ or $\nabla\times\boldsymbol{U}$ is defined such that:

$$\forall\boldsymbol{V}\in\mathbb{R}^3\quad(\operatorname{curl}\boldsymbol{U})\times\boldsymbol{V} = (\underline{\underline{\operatorname{grad}\boldsymbol{U}}} - {}^t\underline{\underline{\operatorname{grad}\boldsymbol{U}}})\cdot\boldsymbol{U} \tag{1.38}$$

In a orthonormal and fix base, the usual relation is found and indeed satisfy the definition of Eq. (1.38) :

$$\operatorname{curl}\boldsymbol{U} \equiv \nabla\times\boldsymbol{U} = \begin{pmatrix} U_{z,y} - U_{y,z} \\ U_{x,z} - U_{z,x} \\ U_{y,x} - U_{x,y} \end{pmatrix} \tag{1.39}$$

It should be noted that the notation $\nabla\times$ can be confusing and the sign ∇ should not be replaced by a “derivative” vector for basis different that fix orthonormal bases. Doing so to perform the computation of the rotational would give incorrect results.

Laplacian The Laplacian of a tensor of order p, noted Δ or ∇^2 is the divergence of the gradient of this tensor, which is also a tensor of order p: By this definition:

$$\Delta\varphi \equiv \nabla^2 \triangleq \operatorname{div}(\operatorname{grad}\varphi) \equiv \nabla\cdot\nabla\varphi \tag{1.40}$$
$$\Delta\boldsymbol{U} \equiv \nabla^2\boldsymbol{U} \triangleq \operatorname{div}(\underline{\underline{\operatorname{grad}\boldsymbol{U}}}) \equiv \nabla\cdot\underline{\underline{\boldsymbol{\nabla U}}} \tag{1.41}$$

In the fix orthonormal base: $\Delta\varphi = \partial_i^2\varphi$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.42}$$

Operator $\boldsymbol{U} \cdot \nabla$ The operator $\boldsymbol{U} \cdot \nabla$ is often used in fluid dynamics instead of the expression $\underline{\underline{\text{grad}}}\, \bullet \cdot \boldsymbol{U}$. It's definition is $U_i \frac{\partial}{\partial x_i}$ using Einstein's convention.

$$(\boldsymbol{U} \cdot \nabla)\varphi = U_i \partial_i \varphi \tag{1.43}$$
$$(\boldsymbol{U} \cdot \nabla)\boldsymbol{V} = U_i \partial_i (V_j \boldsymbol{e_j}) \tag{1.44}$$

In particular, for a fixed cartesian system: $(\boldsymbol{U} \cdot \nabla)\boldsymbol{V} = U_i \partial_i (V_j)\ \boldsymbol{e}_j$

1.4 Vector and field formalism

Differential operations Vector calculus studies various differential operators defined on scalar or vector fields, which are typically expressed in terms of the del operator (∇). The four most important differential operations in vector calculus are: Operation Notation Description Domain/Range Gradient $\text{grad}(f) = \nabla f$ Measures the rate and direction of change in a scalar field. Maps scalar fields to vector fields. Curl $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$ Measures the tendency to rotate about a point in a vector field. Maps vector fields to (pseudo)vector fields. Divergence $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$ Measures the magnitude of a source or sink at a given point in a vector field. Maps vector fields to scalar fields. Laplacian $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$ A composition of the divergence and gradient operations. Maps scalar fields to scalar fields. where the curl and divergence differ because the former uses a cross product and the latter a dot product, and f denotes a scalar field and F denotes a vector field. A quantity called the Jacobian is useful for studying functions when both the domain and range of the function are multivariable, such as a change of variables during integration. [edit] Theorems Likewise, there are several important theorems related to these operators which generalize the fundamental theorem of calculus to higher dimensions: Theorem Statement Description Gradient theorem $\varphi(\mathbf{q}) - \varphi(\mathbf{p}) = \int_{L\,|\,\mathbf{p}\to\mathbf{q}} \nabla\varphi \cdot d\mathbf{r}$The line integral through a gradient (vector) field equals the difference in its scalar field at the endpoints of the curve L. Green's theorem $\iint_{\Sigma\, \in \mathbb{R}^2} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dA = \oint_{\partial\Sigma} (L\, dx + M\, dy)$ The integral of the scalar curl of a vector field over some region in the plane equals the line integral of the vector field over the closed curve bounding the region. Stokes' theorem $\iint_{\Sigma\, \in \mathbb{R}^3} \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r}$ The integral of the curl of a vector field over a surface in $\mathbb{R}^3$ equals the line integral of the vector field over the closed curve bounding the surface. Divergence theorem $\underset{V}{\iiint} (\nabla \cdot \mathbf{F})\, dV = \oiint_{\partial V} \mathbf{F}\ \cdot d\mathbf{S}$ The integral of the divergence of a vector field over some solid equals the integral of the flux through the closed surface bounding the solid.

1.4.1 Fields Operators

Operators definition

$$\text{grad}\, f = \nabla f \quad = \partial_i f_i \qquad \mathbb{R}^n \longmapsto \mathbb{R}^{n+1} \tag{1.45}$$
$$\text{div}\, f = \nabla \cdot f \quad = \qquad \mathbb{R}^n \longmapsto \mathbb{R}^{n-1} \tag{1.46}$$
$$\text{curl}\, f = \nabla \times f = \qquad \mathbb{R}^n \longmapsto \mathbb{R}^{n} \tag{1.47}$$
$$\tag{1.48}$$

1.4.2 Important theorems

Green

Green-Ostrogradski/Divergence theorem For S a closed surface surrounding a volume V and $\boldsymbol{f}$ a piecewise continous and continuously differentiable vector field of any order:

$$\int_{\partial\Omega(t)} \boldsymbol{f} \cdot \boldsymbol{n}\, dS = \int_{\Omega(t)} \mathrm{div}\,(\boldsymbol{f})\, dv \tag{1.49}$$

If a discontinuity surface Σ is present in the volume, Eq. (1.49) is applied on both side of Σ using the notations from Fig. 1.1 where the vector n goes from domain 1 to domain 2. It yields to the general divergence theorem:

$$\int_{\partial\Omega(t)} \boldsymbol{f} \cdot \boldsymbol{n}\, dS = \int_{\Omega(t)} \mathrm{div}\,(\boldsymbol{f})\, dv + \int_{\Sigma} [\![\boldsymbol{f}]\!] \cdot \boldsymbol{n}_{1\to 2}\, d\Sigma \tag{1.50}$$

where $[\![f]\!] = f_2 - f_1$ is the jump of the variable f value across the discontinuity surface Σ. This notation has to be read along with Fig. 1.1 for the convention of the vector $\boldsymbol{n}$ and the two domains.

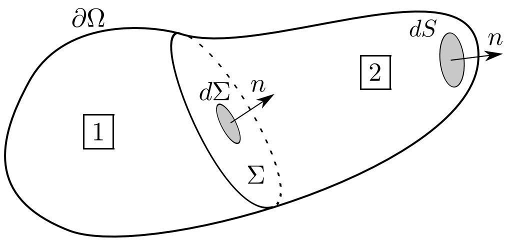

Figure 1.1: Bla bla bla

Discontinuity surface can be: chock waves in supersonic flow, vorticity sheets in shear flow, or boundaries between two non mixing flow. Displacement of this surface within the volume is allowed, and will be further written $\boldsymbol{V_\Sigma}$. A typical example of utilization of Eq. (1.49) would be:

$$\int_{\partial\Omega(t)} \rho \boldsymbol{V}(\boldsymbol{U} \cdot \boldsymbol{n})\, dS = \int_{\Omega(t)} \mathrm{div}\, \rho \boldsymbol{V} \otimes \boldsymbol{U}\, dv + \int_{\Sigma} [\![\rho \boldsymbol{V} \otimes \boldsymbol{U}]\!] \cdot \boldsymbol{n}\, d\Sigma \tag{1.51}$$

A variable $q(M,t)$ such that its field $\mathrm{div}(q\boldsymbol{V}) = 0$ is said to be conservative.

Stokes equation

$$\oint_c \boldsymbol{f} \cdot \boldsymbol{\tau} dl = \int_S \mathrm{curl}\, \boldsymbol{f} \cdot \boldsymbol{n}\, dS \tag{1.52}$$

1.4.3 Relations

Useful cases

$$(\boldsymbol{f} \otimes \boldsymbol{g}) \cdot \boldsymbol{n} = \boldsymbol{f}\,(\boldsymbol{g} \cdot \boldsymbol{n}) \tag{1.53}$$

$$(\underline{\underline{\mathrm{grad}\, \boldsymbol{U}}}) \cdot \boldsymbol{V} = (\underline{\underline{\boldsymbol{\nabla}}}\boldsymbol{U}) \cdot \boldsymbol{V} = (\boldsymbol{V} \cdot \nabla)\boldsymbol{U} \tag{1.54}$$

Integral of a gradient on a closed path - conservative field: Integral on a closed path: For any closed path C, using Stokes theorem and "curl(grad) $\equiv 0$":

$$\oint_C \nabla f \cdot \boldsymbol{dl} = 0 \tag{1.55}$$

A conservative field $\boldsymbol{F}$ is such that its curl is identically zero: $\mathrm{curl}\,\boldsymbol{F} \equiv 0$. It is hence possible to define a scalar potential Φ such that $\boldsymbol{F} = -\nabla\Phi$ Hence from Eq. (1.55):

$$\oint_{C(t)} \boldsymbol{F} \cdot \, \boldsymbol{dl} = 0 \tag{1.56}$$

Useful relations between operators

$$\mathrm{div}(f \otimes \boldsymbol{U}) = \nabla f \cdot \boldsymbol{U} + f\,\mathrm{div}(\boldsymbol{U}) \tag{1.57}$$

$$\mathrm{div}(\rho f \otimes \boldsymbol{U}) = \rho \nabla f \cdot \boldsymbol{U} + f\,\mathrm{div}(\rho \boldsymbol{U}) \tag{1.58}$$

$$\mathrm{curl}(\mathrm{curl}\,\boldsymbol{U}) = \mathrm{grad}(\mathrm{div}\,\boldsymbol{U}) - \nabla^2 \boldsymbol{U} \tag{1.59}$$

$$(\underline{\underline{\mathrm{grad}\,\boldsymbol{U}}}) \cdot \boldsymbol{U} = \mathrm{curl}\,\boldsymbol{U} \times \boldsymbol{U} + \mathrm{grad}\,\frac{U^2}{2} \tag{1.60}$$

Divergence operator:	
$\mathrm{div}(\mathrm{curl}\,\boldsymbol{U}) = 0$	$\nabla \cdot (\nabla \times \mathbf{U}) = 0$
$\mathrm{div}(\mathrm{grad}\,f) = \Delta f$	$\nabla \cdot (\nabla f) = \nabla^2 f$
$\mathrm{div}(\Delta \boldsymbol{U}) = \Delta(\mathrm{div}\,\boldsymbol{U})$	$\nabla \cdot (\nabla^2 \boldsymbol{U}) = \nabla^2(\nabla \cdot \boldsymbol{U})$
$\mathrm{div}\,f\boldsymbol{U} = f\,\mathrm{div}\,\boldsymbol{U} + \boldsymbol{U} \cdot \mathrm{grad}\,f$	$\nabla \cdot (f\boldsymbol{U}) = f\nabla \cdot \boldsymbol{U} + \boldsymbol{U} \cdot \nabla f$
$\mathrm{div}(\boldsymbol{U} \times \boldsymbol{V}) = \mathrm{curl}\,\boldsymbol{U} \cdot \boldsymbol{V} - \boldsymbol{U} \cdot \mathrm{curl}\,\boldsymbol{V}$	$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = (\nabla \times \mathbf{U}) \cdot \mathbf{V} - \mathbf{U} \cdot (\nabla \times \mathbf{V})$
$\mathrm{div}(\underline{\underline{\mathrm{grad}\,\boldsymbol{U}}}) = \Delta \boldsymbol{U}$	$\nabla \cdot (\nabla \boldsymbol{U}) = \nabla^2 \boldsymbol{U}$
$\mathrm{div}({}^t\underline{\underline{\mathrm{grad}\,\boldsymbol{U}}}) = \mathrm{grad}(\mathrm{div}\,\boldsymbol{U})$	$\nabla \cdot^t (\nabla \boldsymbol{U}) = \nabla(\nabla \cdot \boldsymbol{U})$
$\mathrm{div}(\underline{\underline{\boldsymbol{U} \otimes \boldsymbol{V}}}) = \boldsymbol{U}\,\mathrm{div}\,\boldsymbol{V} + \underline{\underline{\mathrm{grad}\,\boldsymbol{U}}} \cdot \boldsymbol{V}$	$\nabla \cdot (\boldsymbol{U} \otimes \boldsymbol{V}) = \boldsymbol{U}\nabla \cdot \boldsymbol{V} + \nabla \boldsymbol{U} \cdot \boldsymbol{V}$
$\mathrm{div}(f\underline{\underline{\boldsymbol{T}}}) = f\,\mathrm{div}\,\underline{\underline{\boldsymbol{T}}} + \underline{\underline{\boldsymbol{T}}} \cdot \mathrm{grad}\,f$	$\nabla \cdot (f\underline{\underline{\boldsymbol{T}}}) = f\nabla \cdot \underline{\underline{\boldsymbol{T}}} + \underline{\underline{\boldsymbol{T}}} \cdot \mathrm{grad}\,f$
$\mathrm{div}(\underline{\underline{\boldsymbol{T}}} \cdot \boldsymbol{U}) = \mathrm{div}^t\underline{\underline{\boldsymbol{T}}} \cdot \boldsymbol{U} + \underline{\underline{\boldsymbol{T}}} : \underline{\underline{\mathrm{grad}\,\boldsymbol{U}}}$	$\nabla \cdot (\underline{\underline{\boldsymbol{T}}} \cdot \boldsymbol{U}) = (\nabla \cdot^t \underline{\underline{\boldsymbol{T}}}) \cdot \boldsymbol{U} + \underline{\underline{\boldsymbol{T}}} : \underline{\underline{\nabla}}\boldsymbol{U}$
$\mathrm{div}(f\underline{\underline{\mathbb{1}}}) = \mathrm{grad}\,f$	$\nabla \cdot f\underline{\underline{\mathbb{1}}} = \nabla f$
Rotational operator:	
$\mathrm{curl}(\mathrm{grad}\,f) = 0$	$\nabla \times (\nabla f) = 0$
$\mathrm{curl}(\mathrm{curl}\,\boldsymbol{U}) = \mathrm{grad}(\mathrm{div}\,\boldsymbol{U}) - \Delta \boldsymbol{U}$	$\nabla \times (\nabla \times \boldsymbol{U}) = \nabla(\nabla \cdot \boldsymbol{U}) - \nabla^2 \boldsymbol{U}$
$\mathrm{curl}(\Delta \boldsymbol{U}) = \Delta(\mathrm{curl}\,\boldsymbol{U})$	$\nabla \times (\nabla^2 \boldsymbol{U}) = \nabla^2(\nabla \times \boldsymbol{U})$
$\mathrm{curl}(f\boldsymbol{U}) = f\,\mathrm{curl}\,\boldsymbol{U} + \mathrm{grad}\,f \times \boldsymbol{U}$	$\nabla \times (f\boldsymbol{U}) = f\nabla \times \boldsymbol{U} + (\nabla f) \times \boldsymbol{U}$
$\mathrm{curl}(\boldsymbol{U} \times \boldsymbol{V}) = \underline{\underline{\mathrm{grad}\,\boldsymbol{U}}} \cdot \boldsymbol{V} - \underline{\underline{\mathrm{grad}\,\boldsymbol{V}}} \cdot \boldsymbol{U} + \boldsymbol{U}\,\mathrm{div}\,\boldsymbol{V} - \boldsymbol{V}\,\mathrm{div}\,\boldsymbol{U}$	$\nabla \times (\boldsymbol{U} \times \boldsymbol{V}) = (\boldsymbol{V} \cdot \nabla)\boldsymbol{U} - (\boldsymbol{U} \cdot \nabla)\boldsymbol{V} + (\nabla \cdot \boldsymbol{V})\boldsymbol{U} - (\nabla \cdot \boldsymbol{U})\boldsymbol{V}$
Gradient operator:	
$\mathrm{grad}(fg) = f\,\mathrm{grad}\,g + g\,\mathrm{grad}\,f$	$\nabla(fg) = f\nabla g + g\nabla f$
$\mathrm{grad}(\boldsymbol{U} \cdot \boldsymbol{V}) = \underline{\underline{\mathrm{grad}\,\boldsymbol{U}}} \cdot \boldsymbol{V} + \underline{\underline{\mathrm{grad}\,\boldsymbol{V}}} \cdot \boldsymbol{U} + \boldsymbol{U} \times \mathrm{curl}\,\boldsymbol{V} + \boldsymbol{V} \times \mathrm{curl}\,\boldsymbol{U}$	$\nabla(\boldsymbol{U} \cdot \boldsymbol{V}) = (\boldsymbol{V} \cdot \nabla)\boldsymbol{U} \cdot +(\boldsymbol{U} \cdot \nabla)\boldsymbol{V} + \boldsymbol{U} \times (\nabla \times \boldsymbol{V}) + \boldsymbol{V} \times (\nabla \times \boldsymbol{U})$
$(\underline{\underline{\mathrm{grad}\,\boldsymbol{U}}}) \cdot \boldsymbol{U} = \mathrm{curl}\,\boldsymbol{U} \times \boldsymbol{U} + \mathrm{grad}\,\frac{U^2}{2}$	$(\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = (\nabla \times \boldsymbol{U}) \times \boldsymbol{U} + \nabla \frac{U^2}{2}$
$\underline{\underline{\mathrm{grad}\,f\boldsymbol{V}}} = f\underline{\underline{\mathrm{grad}\,\boldsymbol{V}}} + \boldsymbol{V} \cdot^t \mathrm{grad}\,f$	